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On the convergence properties of weakly multiplicative systems

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To my teacher Professor K. Tandori on his 50th birthday

§ 1. Results

In this paper $(X, \mathcal{A}, \mu)$ will be a measure space with a σ-finite [1]) non-negative measure μ, unless otherwise stated. Let $\{\varphi_i\}$ be a system of measurable functions defined on X and taking on real values. The crucial property of the system $\{\varphi_i\}$ which will be used in the proofs is the fact that it is "weakly multiplicative" in the sense that the integrals $\int \varphi_{i_1}\varphi_{i_2}\dots\varphi_{i_r}\,d\mu$ [2]) are small if $i_1, i_2, \dots, i_r$ are different integers for a fixed even integer r, $r \geqq 4$. More exactly, set

$$\beta_{i_1, i_2, \dots, i_r} = \int \varphi_{i_1}\varphi_{i_2}\dots\varphi_{i_r}\,d\mu$$

and denote by B_r the infinite vector whose components are $\beta_{i_1, i_2, \dots, i_r}$, where $i_1, i_2, \dots, i_r$ simultaneously run over the integers satisfying only the condition $1 \leqq i_1 < i_2 < \dots < i_r$. The notion of *weak multiplicity* is understood in the sense that the symmetric and absolute norm of B_r in l_q is finite:

$$\|B_r\|_q = \Big[\sum_{1 \leqq i_1 < i_2 < \dots < i_r} |\beta_{i_1, i_2, \dots, i_r}|^q\Big]^{1/q} < \infty,$$

where q is a fixed number, $1 \leqq q < \infty$. The purpose of the present paper is to obtain

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[1]) If the measure μ is not σ-finite, then instead of the original measure space $(X, \mathcal{A}, \mu)$ consider its restriction to the union X_1 of the supports of the integrable functions φ_i $(i=1, 2, \dots)$ of the system in question. It is clear that μ is σ-finite on X_1 and concerning the problem of convergence the set $X \setminus X_1$ is irrelevant.

[2]) For the sake of simplicity we do not indicate the arguments of functions; we write φ, f etc. instead of $\varphi(x), f(x)$, etc., unless this causes any confusion; we write $\int \varphi\,d\mu$ and L_r instead of $\int_X \varphi\,d\mu$ and $L_r(X, \mathcal{A}, \mu)$, respectively; we also say "almost everywhere" (in abbreviation: a.e.) instead of "μ-almost everywhere".

somewhat stronger results than those of GAPOŠKIN [6], KOMLÓS and RÉVÉSZ [9] under less restrictive conditions.

Throughout the paper r will denote an even integer, $r \geqq 4$, p will denote a real number, $1 < p \leqq 2$, while q will denote the "complementary" exponent, i.e., $1/p + 1/q = 1$. Besides them, C, C_r, $C_{r,p}$, $C^*_{r,p}$, etc. will denote positive constants, not necessarily the same at each occurrence. Furthermore, K, K_1, and K_2 will denote positive numbers, which are (upper or lower) bounds of the integrals of the appropriate power of functions φ_i in question.

We recall here the well-known notion of $\mathscr{S}_r$ system [7, pp. 243—246]: a system $\{\varphi_i\}$ belonging to L_r is said to be an $\mathscr{S}_r$ *system* if for every sequence $\{c_i\}$ of real numbers and for every positive integer n the inequality

$$\int \left(\sum_{i=1}^{n} c_i \varphi_i \right)^r d\mu \leqq C_r \left(\sum_{i=1}^{n} c_i^2 \right)^{r/2} \tag{1.1}$$

holds.[3]) Let us introduce the following generalization of this notion. We say that $\{\varphi_i\}$ is an $\mathscr{S}_{r,p}$ *system* if for every sequence $\{c_i\}$ and for every integer n we have

$$\int \left(\sum_{i=1}^{n} c_i \varphi_i \right)^r d\mu \leqq C_{r,p} \left(\sum_{i=1}^{n} |c_i|^p \right)^{r/p}.$$

In the study of the convergence of series $\sum c_i \varphi_i$, where $\{\varphi_i\}$ is a weakly multiplicative system ("*direct theorems*") a result of Erdős—Stečkin (as far the proof, see GAPOŠKIN [4, pp. 28—31]) and its generalization, due to TJURNPÜ [15], play a key role: *If* $\{\varphi_i\}$ *is an* $\mathscr{S}_{r,p}$ *system and if* $r > p > 1$, *then there exists another constant* $C^*_{r,p}$ *such that for every sequence* $\{c_i\}$ *and for every integer n the inequality*

$$\int \max_{1 \leqq k \leqq n} \left(\sum_{i=1}^{k} c_i \varphi_i \right)^r d\mu \leqq C^*_{r,p} \left(\sum_{i=1}^{n} |c_i|^p \right)^{r/p} \tag{1.2}$$

also holds true.[4])

Making use of this result we can arrive in a routine way at the following assertion: *Every* $\mathscr{S}_{r,p}$ *system is an unconditional convergence system* (UCS) *for* l_p *if* $r > p > 1$. This means that every series $\sum c_i \varphi_i$ with $\sum |c_i|^p < \infty$ is convergent a.e. in every arrangement of its terms. Furthermore, (1.2) yields also the slightly stronger assertion that, under the above conditions, the maximum of the moduli of the partial sums of $\sum c_i \varphi_i$ belongs to L_r in every arrangement of the terms.

Our main direct theorem reads as follows.

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[3]) The notion of $\mathscr{S}_r$ system is defined for any positive number r, but when r is not an even integer, on the left-hand side of (1.1) we must have $\int \left| \sum_{i=1}^{n} c_i \varphi_i \right|^r d\mu$.

[4]) The case $p = 2$ is due to Erdős ($r = 4$) and Stečkin ($r > 2$), while the general case $r > p > 1$ was treated by Tjurnpü.

Theorem 1. *Let r be an even integer, $r \geqq 4$, let p be a real number, $1<p\leqq 2$, and let q be defined by $1/p+1/q=1$. Let $\{\varphi_i\}$ be a system of functions in L_r for which*

$$\int \varphi_i^r \, d\mu \leqq K \quad (i = 1, 2, \ldots) \tag{1.3}$$

and

$$\|B_r\|_q^q = \sum_{1 \leqq i_1 < i_2 < \ldots < i_r} \left| \int \varphi_{i_1} \varphi_{i_2} \ldots \varphi_{i_r} \, d\mu \right|^q < \infty. \tag{1.4}$$

Then $\{\varphi_i\}$ is an $\mathcal{S}_{r,p}$ system.

Consequently, $\{\varphi_i\}$ is an UCS *for l_p and the maximum of the moduli of the partial sums of $\sum c_i \varphi_i$ with $\sum |c_i|^p < \infty$ belongs to L_r in every arrangement of the terms.*

We point out that in Theorem 1 the stipulation on p is essential. In other words, if condition (1.4) is required to hold for a q such that $1<q<2$, this stronger condition does not imply the a.e. convergence of $\sum c_i \varphi_i$ for any $\{c_i\} \in l_p \setminus l_2$ in the case when $p>2$. The reason is that the converse of Theorem 1, under a natural further assumption on the lower boundedness of $\int \varphi_i^2 d\mu$ $(i=1, 2, \ldots)$, is also true. If the series $\sum c_i \varphi_i$ converges at the points of a set of positive measure, then $\sum c_i^2$ is finite. We shall prove much more general theorems, too.

In the sequel we restrict ourselves to the case $r=4$. This case illuminates the general situation well enough.

In the study of the divergence of series $\sum c_i \varphi_i$, where $\{\varphi_i\}$ is a weakly multiplicative system ("*converse theorems*") the following inequality if of basic importance.

Theorem 2. *Let $\{\varphi_i\}$ be a system of functions in L_4 for which*

$$\int \varphi_i^4 \, d\mu \leqq K \quad (i = 1, 2, \ldots), \tag{1.5}$$

$$\|B_4\|_2^2 = \sum_{1 \leqq i < k < l < m} \left(\int \varphi_i \varphi_k \varphi_l \varphi_m \, d\mu \right)^2 < \infty, \tag{1.6}$$

and

$$K_1 \leqq \int_F \varphi_i^2 \, d\mu \leqq K_2 \quad (i > i_0), \tag{1.7}$$

where F is a set of positive and finite measure, and let δ be a positive number. Then there exists an integer n_0 such that for any sequence $\{c_i\}$ of numbers and for any integer $n \geqq n_0$ we have

$$(1-\delta) K_1 \sum_{i=n_0}^{n} c_i^2 \leqq \int_F \left(\sum_{i=n_0}^{n} c_i \varphi_i \right)^2 d\mu \leqq (1+\delta) K_2 \sum_{i=n_0}^{n} c_i^2. \tag{1.8}$$

We note that the second inequality of (1.7) is a consequence of (1.5) with $K_2 = [K\mu(F)]^{1/2}$, because of $\mu(F) < \infty$.

We shall consider an arbitrary linear method of summation defined by a doubly infinite matrix $T^* = (\alpha_{mn})$, whose elements satisfy the first and third conditions of regularity:[5])

(1.9) $$\lim_{m\to\infty} \alpha_{mn} = 0 \quad (n = 1, 2, \ldots)$$

and

(1.10) $$\lim_{m\to\infty} \sum_{n=1}^{\infty} \alpha_{mn} = 1.$$

All linear methods of summation used in analysis are T^* methods. Set

$$t_m = \sum_{n=1}^{\infty} \alpha_{mn} s_n, \quad s_n = \sum_{i=1}^{n} c_i \varphi_i.$$

We say that the series $\sum c_i \varphi_i$ is T^* *summable* to a limit s if the T^* *mean* t_m tends to s as $m \to \infty$.

Theorem 3. *Let* $\{\varphi_i\}$ *be a system of functions in* L_4 *satisfying conditions* (1,5), (1.6), *and*

(1.11) $$\liminf_{i\to\infty} \int_E \varphi_i^2 \, d\mu > 0,$$

where E is a set of positive measure. If a series $\sum c_i \varphi_i$ *is* T^* *summable or, more generally, its* T^* *means are bounded on E, then* $\sum c_i^2$ *is finite.*

The following proposition immediately follows from Theorems 1 and 3.

Corollary 1. *If the system* $\{\varphi_i\}$ *satisfies* (1.5), (1.6), *and* (1.11) *holds for every set E of positive measure, then any series* $\sum c_i \varphi_i$ *is a.e. convergent or a.e. not* T^* *summable in any arrangement of its terms, according as the series* $\sum c_i^2$ *is finite or not.*

In probability theory this fact is called the *law of zero or unity.*

For certain problems it is desirable to have a similar result in the case, when only one-sided boundedness of the T^* means is supposed. Before stating our next result in an explicit form, we introduce the following notation. Set

$$R_{mi} = \sum_{n=i}^{\infty} \alpha_{mn} \quad (i = 1, 2, \ldots).$$

It is obvious that the mean t_m can be rewritten into the form

$$t_m = \sum_{n=1}^{\infty} \alpha_{mn} s_n = \sum_{i=1}^{\infty} R_{mi} c_i \varphi_i.$$

[5]) The second condition of regularity, which is neglected in our paper reads as follows: the sums $\sum_{n=1}^{\infty} |\alpha_{mn}|$ are bounded ($m = 1, 2, \ldots$). As to the notion of regularity, see, e.g., Zygmund [16, p. 74].

It can be easily seen from (1.9) and (1.10) that

$$\lim_{m\to\infty} R_{mi} = 1 \quad (i = 1, 2, \ldots). \tag{1.12}$$

Theorem 4. *Let $\{\varphi_i\}$ be a system of functions in L_4 satisfying conditions* (1.5) *and* (1.6); *furthermore, assume that* (1.11) *holds for every set E of positive measure. If $\sum c_i^2$ is not finite, then the set of points x at which* [6])

$$\lim_{m\to\infty} \frac{t_m^+(x)}{\left[\sum_{i=1}^{\infty} R_{mi}^2 c_i^2\right]^{1/2}} = 0 \tag{1.13}$$

holds, is of measure zero.

We remark that the sum in brackets is finite by virtue of Theorem 3 provided that the series defining $t_m(x)$ converges on a set of positive measure. From (1.12) it follows immediately that the denominator of (1.13) tends to ∞ as $m\to\infty$. Hence Theorem 4 implies

Corollary 2. *Under the conditions of Theorem* 4, *and if the T^* means of $\sum c_i\varphi_i$ are bounded from above (or from below) on a set of positive measure, then $\sum c_i^2$ is finite.*

§ 2. Historical comments

Let $\{\varphi_i\}$ be a system of measurable functions on $(X, \mathcal{A}, \mu)$, $\mu(X)<\infty$, such that $\varphi_i\in L_q$ for every $q\geqq 2$, or, in particular, let φ_i be essentially bounded $(i=1, 2, \ldots)$. In this section we assume that

$$\int \varphi_i\,d\mu = 0 \quad \text{and} \quad \int \varphi_i^2\,d\mu = 1 \quad (i = 1, 2, \ldots). \tag{2.1}$$

The following definitions [7]) were introduced by ALEXITS [1, pp. 186—187]: $\{\varphi_i\}$ is said to be

(i) a *multiplicative system* (MS) if

$$\int \varphi_{i_1}\varphi_{i_2}\ldots\varphi_{i_k}\,d\mu = 0;$$

(ii) a *strongly multiplicative system* (SMS) if

$$\int \varphi_{i_1}^{\alpha_1}\varphi_{i_2}^{\alpha_2}\ldots\varphi_{i_k}^{\alpha_k}\,d\mu = 0,$$

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[6]) Here $t_m^+=\max(0, t_m)$.

[7]) In earlier papers the underlying measure space $(X, \mathcal{A}, \mu)$ was a special probability space: $X=[0,1]$, $\mathcal{A}$ is the class of the Borel subsets of $[0,1]$, and μ is the Lebesgue measure on it.

where $\alpha_1, \alpha_2, \ldots, \alpha_k$ can be equal to 1 or 2 but at least one of them is equal to 1;

(iii) an *equinormed strongly multiplicative system* (ESMS) if

$$\int \varphi_{i_1}^{\alpha_1} \varphi_{i_2}^{\alpha_2} \ldots \varphi_{i_k}^{\alpha_k}\, d\mu = \int \varphi_{i_1}^{\alpha_1}\, d\mu \int \varphi_{i_2}^{\alpha_2}\, d\mu \ldots \int \varphi_{i_k}^{\alpha_k}\, d\mu,$$

where $\alpha_1, \alpha_2, \ldots, \alpha_k$ can be equal to 1 or 2. In all these three definitions: $1 \leqq i_1 < i_2 < \ldots < i_k$, $k=2, 3, \ldots$.

Making use of the method of the Lebesgue functions, Alexits [1a] (see also Alexits and Tandori [3]) proved the following

Theorem A. *If $\{\varphi_i\}$ is a uniformly bounded* ESMS, *then the condition*

$$\sum_{i=1}^{\infty} c_i^2 < \infty \tag{2.2}$$

implies the a.e. convergence of the series

$$\sum_{i=1}^{\infty} c_i \varphi_i. \tag{2.3}$$

Later Alexits and Sharma [2] showed that Theorem A remains valid in the case when $\{\varphi_i\}$ is only a uniformly bounded MS. A simpler proof of this assertion was found by Preston [12].

Obviously any independent system of random variables defined on a probability space $(X, \mathcal{A}, \mu)$ and satisfying (2.1) is an ESMS. A classical Kolmogorov theorem states that if the random variables $\varphi_1, \varphi_2, \ldots$ are independent with expectation 0 and variance 1, then condition (2.2) implies the a.e. convergence of (2.3). Therefore, even the theorem of Alexits and Tandori would be much stronger than Kolmogorov's theorem if the condition of uniform boundedness could be dropped.

The first step toward this direction was made by Révész [13].

Theorem B. *Suppose that*

$$\int \varphi_i^4\, d\mu \leqq K \quad (i = 1, 2, \ldots) \tag{2.4}$$

and

$$\int \varphi_i^2 \varphi_k \varphi_l\, d\mu = \int \varphi_i^2 \varphi_k\, d\mu = \int \varphi_i \varphi_k \varphi_l \varphi_m\, d\mu = \int \varphi_i \varphi_k \varphi_l\, d\mu = \int \varphi_i \varphi_k\, d\mu = 0,$$

where i, k, l, m are different integers. Furthermore, let $\{c_i\}$ be a sequence for which there exists an integer s such that

$$\sum_{i=1}^{\infty} c_i^2 l_s^2(i) < \infty, \tag{2.5}$$

where $l_s(i)$ means the sth iterate of $\log i$.[8]) *Then the series* (2.3) *converges a.e.*.

[8]) I.e., $l_s(t)$ is defined by the following recurrence relation: $l_s(t) = l(l_{s-1}(t))$ if $s \geqq 2$, where $l(t) = l_1(t) = \log t$ if $t \geqq 2$, and $=1$ if $0 < t < 2$.

Condition (2.5) is not very far from condition (2.2). This fact suggested the conjecture that (2.5) can be replaced by (2.2). This was shown by GAPOŠKIN [5], under weaker assimptions on $\{\varphi_i\}$.

Theorem C. *Suppose that condition* (2.4),

$$\int \varphi_i \varphi_k \varphi_l \varphi_m \, d\mu = 0, \tag{2.6}$$

and

$$\int \varphi_i^2 \varphi_k \varphi_l \, d\mu = 0 \tag{2.7}$$

hold, where i, k, l, m are different integers. Then $\{\varphi_i\}$ is an $\mathcal{S}_4$ system.

KOMLÓS and RÉVÉSZ [9] observed that condition (2.7) can be omitted.

Theorem D. *Under conditions* (2.4) *and* (2.6), *$\{\varphi_i\}$ is an $\mathcal{S}_4$ system.*

We note that this fact was essentially formulated previously by SERFLING [14], but we think his proof is not complete. At the same time, independently of the above authors, GAPOŠKIN [6] also obtained similar results.

Theorem E. *If*

$$\int \varphi_i^r \, d\mu \leqq K \quad (i = 1, 2, \ldots) \tag{2.8}$$

and

$$\int \varphi_{i_1} \varphi_{i_2} \ldots \varphi_{i_r} \, d\mu = 0 \quad (1 \leqq i_1 < i_2 < \ldots < i_r), \tag{2.9}$$

where r is an even integer, $r \geqq 4$, then $\{\varphi_i\}$ is an $\mathcal{S}_r$ system.

In addition, Gapoškin pointed out that the vanishing of the integrals in (2.9) is of no relevance, only their "relative smallness" is needed.

Theorem F. *Suppose that* (2.8) *holds and there exists a non-negative function $f(i)$ $(i=1, 2, \ldots)$ such that*

$$\left| \int \varphi_{i_1} \varphi_{i_2} \ldots \varphi_{i_r} \, d\mu \right| \leqq \min \{ f(i_2 - i_1), f(i_4 - i_3), \ldots, f(i_r - i_{r-1}) \}$$

for every $1 \leqq i_1 < i_2 < \ldots < i_r$ and

$$\sum_{i=1}^{\infty} i^{(r-2)/2} f(i) < \infty, \tag{2.10}$$

where r is an even integer, $r \geqq 4$, then $\{\varphi_i\}$ is an $\mathcal{S}_r$ system.

We mention that in [9] KOMLÓS and RÉVÉSZ also stated this result for $r=4$.

Our Theorem 1 evidently contains Theorem D and Theorem E even in the special case $p=2$. Theorem 1 and Theorem F are incomparable, as no one of the conditions (1.4) and (2.10) implies the other.

Inequality (1.1) expressing the $\mathcal{S}_r$ property of a system is valid for a large class of independent random variables and is a classical result of probability theory. Furthermore, it is well-known for lacunary trigonometric series[9]) (cf. [16, p. 215]). In the case of multiplicative systems, inequality (1.1) was proved first by the present author [10].

Theorem G. *Let* $\{\varphi_i\}$ *be a uniformly bounded* SMS *and let* q *be any positive number. Then for every sequence* $\{c_i\}$ *and for every integer* n *we have*

$$C_q'\left(\sum_{i=1}^n c_i^2\right)^{q/2} \leqq \int_0^1 \left|\sum_{i=1}^n c_i\varphi_i\right|^q d\mu = C_q\left(\sum_{i=1}^n c_i^2\right)^{q/2}.$$

Now we provide a brief review on the converse results. The first such result is also due to ALEXITS [1, p. 194].

Theorem H.[10]) *Suppose that conditions* (2.6), (2.7), *and*

$$\int \varphi_i^2\varphi_k^2\, d\mu = 1 \tag{2.11}$$

are satisfied, where i *and* k *are different integers, furthermore, for every set* E *of positive measure the relation*

$$\int_E \varphi_i^2\, d\mu \geqq K_1\mu(E) \quad (i > i_0) \tag{2.12}$$

holds. If the series (2.3) *is summable on a set of positive measure by a regular summation method that is finite with respect to the rows, then its coefficients satisfy condition* (2.2).

Later ALEXITS and SHARMA [2] showed that Theorem H is true if condition (2.11) is replaced by the condition of uniform boundedness of $\{\varphi_i\}$.

The present author proved [11] that if (2.4) holds, then condition (2.11) yields (2.12) with a constant $K_1\sim 1$. More precisely, our result reads as follows.

Theorem I. *Suppose we are given a set* E *of positive measure and a positive number* δ. *Under conditions* (2.4), (2.6), (2.7), *and* (2.11) *there exists an integer* n_0 *such that for any sequence* $\{c_i\}$ *and for any integer* $n\geqq n_0$ *we have*

$$(1-\delta)\mu(E)\sum_{i=n_0}^n c_i^2 \leqq \int_E \left(\sum_{i=n_0}^n c_i\varphi_i\right)^2 d\mu \leqq (1+\delta(\mu(E)\sum_{i=n_0}^n c_i^2.$$

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[9]) The trigonometric series $\sum_{k=1}^{\infty}(a_k\cos n_k t + b_k \sin n_k t)$ is said to be lacunary if n_k's are integers and $n_{k+1}/n_k \geqq q > 1$ $(k=1, 2, \ldots)$.

[10]) Here we give the original theorem of Alexits with a slight modification. It is evident from his proof that this modification also holds true. This remark relates also to Theorem I.

Furthermore, if the T^* means of the series (2.3) *are bounded on a set of positive measure, then condition* (2.2) *holds.*

KOMLÓS [8] observed that conditions (2.7) and (2.11) are superfluous.

Theorem J. *Suppose that $\{\varphi_i\}$ satisfies conditions* (2.4), (2.6), *and*

$$\liminf_{i\to\infty}\int_E \varphi_i^2\,d\mu>0$$

for every set E of positive measure, then the convergence of (2.3) *on any set of positive measure implies* (2.2).

Obviously, Theorem 3 contains Theorem J even in the special case of convergence, and Theorem 4 is a generalization of a result of ZYGMUND [16, p. 205]. We note that an intermediate step of generalization of Zygmund's theorem referred to above appeared in [11].

We remark that all the theorems mentioned, except Theorem J, was originally stated for finite measure spaces, in spite of the fact that finiteness is essential only in the proof of Theorem A.

§ 3. Proof of Theorem 1

The following lemma is of fundamental significance in establishing direct theorems of convergence.

Lemma 1. *Let $a_1, a_2, \dots, a_n$ be real numbers, let r be an integer, $r\geqq 2$, and let p be a positive real number, $p\leqq 2$. Set*

$$S=\sum_{i=1}^n a_i,\quad S_p=\left(\sum_{i=1}^n |a_i|^p\right)^{1/p},$$

and

$$T_r=\sum_{1\leqq i_1<i_2<\dots<i_r\leqq n} a_{i_1}a_{i_2}\dots a_{i_r}.$$

Then

$$|S^r-r!\,T_r|\leqq C_r\{S_p^r+S_p|S|^{r-1}\}.$$

This lemma immediately follows from that of GAPOŠKIN [6] if we take into consideration that

$$S_2\leqq S_p\quad (0<p\leqq 2)$$

and that for any positive numbers a and b the inequality

$$a^{r-1}b+a^{r-2}b^2+\dots+a^2b^{r-2}\leqq\frac{1}{2}(r-2)(a^r+ab^{r-1})$$

holds.

Proof of Theorem 1. By virtue of Lemma 1 we have

$$\int S^r\,d\mu \leqq C_r\left\{\int S_p^r\,d\mu+\int S_p|S|^{r-1}\,d\mu\right\}+r!\left|\int T_r\,d\mu\right|,$$

where A, S_p, and T_r are defined as follows:

$$S=\sum_{i=1}^{n} c_i\varphi_i,\quad S_p=\left(\sum_{i=1}^{n}|c_i\varphi_i|^p\right)^{1/p},$$

and

$$T_r=\sum_{1\leqq i_1<i_2<\ldots<i_r\leqq n} c_{i_1}c_{i_2}\ldots c_{i_r}\int \varphi_{i_1}\varphi_{i_2}\ldots\varphi_{i_r}\,d\mu.$$

Using Minkowski's inequality and (1.3), we obtain:

$$\int S_p^r\,d\mu=\int\left(\sum_{i=1}^{n}|c_i\varphi_i|^p\right)^{r/p}d\mu\leqq\left[\sum_{i=1}^{n}\left(\int|c_i\varphi_i|^r\,d\mu\right)^{p/r}\right]^{r/p}\leqq$$

$$\leqq\left[\sum_{i=1}^{n}|c_i|^p\left(\int\varphi_i^r\,d\mu\right)^{p/r}\right]^{r/p}\leqq K\left(\sum_{i=1}^{n}|c_i|^p\right)^{r/p}.$$

Now Hölder's inequality gives that

$$\int S_p|S|^{r-1}\,d\mu\leqq\left(\int S_p^r\,d\mu\right)^{1/r}\left(\int S^r\,d\mu\right)^{(r-1)/r}\leqq K^{1/r}\left(\sum_{i=1}^{n}|c_i|^p\right)^{1/p}\left(\int S^r\,d\mu\right)^{(r-1)/r}.$$

Finally, we can estimate $\left|\int T_r d\mu\right|$ in the following way:

$$\left|\int T_r\,d\mu\right|\leqq\sum_{1\leqq i_1<\ldots<i_r\leqq n}\left|c_{i_1}\ldots c_{i_r}\int\varphi_{i_1}\ldots\varphi_{i_r}\,d\mu\right|\leqq$$

$$\leqq\Big(\sum_{1\leqq i_1<\ldots<i_r\leqq n}|c_{i_1}|^p\ldots|c_{i_r}|^p\Big)^{1/p}\Big(\sum_{1\leqq i_1<\ldots<i_r\leqq n}\left|\int\varphi_{i_1}\ldots\varphi_{i_r}\,d\mu\right|^q\Big)^{1/q}\leqq$$

$$\leqq\|B_r\|_q\left(\sum_{i=1}^{n}|c_i|^p\right)^{r/p}.$$

Putting this all together we obtain:

$$\int S^r\,d\mu\leqq C_r\left\{K\left(\sum_{i=1}^{n}|c_i|^p\right)^{r/p}+K^{1/r}\left(\sum_{i=1}^{n}|c_i|^p\right)^{r/p}\left(\int S^r\,d\mu\right)^{(r-1)/r}\right\}+$$

$$+r!\,\|B_r\|_q\left(\sum_{i=1}^{n}|c_i|^p\right)^{r/p}.$$

Setting

$$z=\frac{\left(\int S^r\,d\mu\right)^{1/r}}{\left(\sum_{i=1}^{n}|c_i|^p\right)^{1/p}},$$

provided that $\sum|c_i|^p\neq0$, we arrive at the inequality

$$z^r\leqq C_r(K+K^{1/r}z^{r-1})+r!\,\|B_r\|_q.$$

Using the elementary fact that if for positive z, a, and b we have

$$z^r \leqq a z^{r-1} + b$$

then

$$z \leqq a + b^{1/r},$$

we get the desired inequality

$$\left(\int S^r \, d\mu\right)^{1/r} \leqq \{C_r K^{1/r} + (C_r K + r! \, \|B_r\|_q)^{1/r}\} \left(\sum_{i=1}^{n} |c_i|^p\right)^{1/p},$$

which expresses the $\mathscr{S}_{r,p}$ property of the system $\{\varphi_i\}$. Thus Theorem 1 is proved.

§ 4. Proof of two lemmas

We begin with proving a Bessel type inequality for weakly multiplicative systems. We consider the generalized Fourier coefficients of a function f in L_2 with respect to the system $\{\varphi_i \varphi_k\}$, defined as follows:

$$\gamma_{ik} = \int f \varphi_i \varphi_k \, d\mu \quad (i, k = 1, 2, \ldots; \; i \neq k). \tag{4.1}$$

Lemma 2. *Let $\{\varphi_i\}$ be a system of functions in L_4 satisfying conditions* (1.5) *and* (1.6). *Then for any square integrable function f we have*

$$\sum_{1 \leqq i < k} \gamma_{ik}^2 \leqq C \int f^2 \, d\mu. \tag{4.2}$$

Proof of Lemma 2. The proof is similar to that of Bessel's inequality, well-known in the theory of orthogonal series. We note that this lemma has already been formulated and proved by Komlós [8] under more restricted conditions.

We shall use the elementary identity

$$\Big(\sum_{1 \leqq i < k \leqq n} a_{ik}\Big)^2 = \sum_{i=1}^{n} \sum_{\substack{k=1 \\ k \neq i}}^{n} \sum_{\substack{l=i \\ l \neq i}}^{n} a_{ik} a_{il} - \sum_{1 \leqq i < k \leqq n} a_{ik}^2 + 2 \sum_{1 \leqq i < k < l < m \leqq n} (a_{ik} a_{lm} + a_{il} a_{km} + a_{im} a_{kl}).$$

Setting $a_{ik} = \gamma_{ik} \varphi_i \varphi_k$ and taking into account (4.1), we obtain the inequality

$$\begin{aligned} 0 \leqq \int \Big(\lambda f - \sum_{1 \leqq i < k \leqq n} \gamma_{ik} \varphi_i \varphi_k\Big)^2 d\mu = \lambda^2 \int f^2 \, d\mu - 2\lambda \sum_{1 \leqq i < k \leqq n} \gamma_{ik}^2 + \\ + \sum_{i=1}^{n} \sum_{\substack{k=1 \\ k \neq i}}^{n} \sum_{\substack{l=1 \\ l \neq i}}^{n} \gamma_{ik} \gamma_{il} \int \varphi_i^2 \varphi_k \varphi_l \, d\mu - \sum_{1 \leqq i < k \leqq n} \gamma_{ik}^2 \int \varphi_i^2 \varphi_k^2 \, d\mu + \\ + 2 \sum_{1 \leqq i < k < l < m \leqq n} (\gamma_{ik} \gamma_{lm} + \gamma_{il} \gamma_{km} + \gamma_{im} \gamma_{kl}) \int \varphi_i \varphi_k \varphi_l \varphi_m \, d\mu, \end{aligned} \tag{4.3}$$

where λ denotes a parameter, whose value will be determined later, and n is fixed for temporarily.

Consider separately the third and the fifth sum on the right-hand side of (4.3). We remind that, under the conditions of Lemma 2, $\{\varphi_i\}$ is an $\mathscr{S}_4$ system $(p=2)$, by virtue of Theorem 1. Using the Buniakowski—Schwarz inequality, condition (1.5), and the $\mathscr{S}_4$ property of $\{\varphi_i\}$, we obtain that

$$(4.4)\qquad S_1=\sum_{i=1}^{n}\sum_{\substack{k=1\\k\neq i}}^{n}\sum_{\substack{l=1\\l\neq i}}^{n}\gamma_{ik}\gamma_{il}\int\varphi_i^2\varphi_k\varphi_l\,d\mu=\sum_{i=1}^{n}\int\varphi_i^2\left(\sum_{\substack{k=1\\k\neq i}}^{n}\gamma_{ik}\,\varphi_k\right)^2d\mu\leqq$$

$$\leqq\sum_{i=1}^{n}\left[\int\varphi_i^4\,d\mu\right]^{1/2}\left[\int\left(\sum_{\substack{k=1\\k\neq i}}^{n}\gamma_{ik}\varphi_k\right)^4d\mu\right]^{1/2}\leqq$$

$$\leqq K^{1/2}C_4^{1/2}\sum_{i=1}^{n}\sum_{\substack{k=1\\k\neq i}}^{n}\gamma_{ik}^2=2K^{1/2}C_4^{1/2}\sum_{1\leqq i<k\leqq n}\gamma_{ik}^2.$$

Now applying the Cauchy inequality, from (1.6) it follows that

$$\sum_{1\leqq i<k<l<m\leqq n}\gamma_{ik}\gamma_{lm}\int\varphi_i\varphi_k\varphi_l\varphi_m\,d\mu\leqq\left[\sum\gamma_{ik}^2\gamma_{lm}^2\right]^{1/2}\left[\sum\left(\int\varphi_i\varphi_k\varphi_l\varphi_m\,d\mu\right)^2\right]^{1/2}\leqq$$

$$\leqq\|B_4\|_2\left[\sum_{1\leqq i<k<l<m\leqq n}\gamma_{ik}^2\gamma_{lm}^2\right]^{1/2}\leqq\|B_4\|_2\sum_{1\leqq i<k\leqq n}\gamma_{ik}^2.$$

Hence we find that

$$(4.5)\qquad S_2=2\sum_{1\leqq i<k<l<m\leqq n}(\gamma_{ik}\gamma_{lm}+\gamma_{il}\gamma_{km}+\gamma_{im}\gamma_{kl})\int\varphi_i\varphi_k\varphi_l\varphi_m\,d\mu\leqq$$

$$\leqq 6\|B_4\|_2\sum_{1\leqq i<k\leqq n}\gamma_{ik}^2.$$

Estimating the right-hand side of (4.3) by means of inequalities (4.4) and (4.5), we arrive at

$$0\leqq\lambda^2\int f^2\,d\mu-2\lambda\sum_{1\leqq i<k\leqq n}\gamma_{ik}^2+S_1-\sum_{1\leqq i<k\leqq n}\gamma_{ik}^2\int\varphi_i^2\varphi_k^2\,d\mu+S_2\leqq$$

$$\leqq\lambda^2\int f^2\,d\mu-2(\lambda-K^{1/2}C_4^{1/2}-3\|B_4\|_2)\sum_{1\leqq i<k\leqq n}\gamma_{ik}^2,$$

where the fourth sum on the right-hand side was simply omitted, being always non-negative. Choosing

$$\lambda=2(K^{1/2}C_4^{1/2}+3\|B_4\|_2),$$

we get that

$$\sum_{1\leqq i<k\leqq n}\gamma_{ik}^2\leqq\lambda\int f^2\,d\mu=2(K^{1/2}C_4^{1/2}+3\|B_4\|_2)\int f^2\,d\mu.$$

Since this is true for all n, the assertion of Lemma 2 follows.

In the proof of Theorem 4 we need

Lemma 3. *Let $\{\varphi_i\}$ be a system of functions in L_4 satisfying conditions* (1.5) *and* (1.6), *and let F be a measurable set of finite measure. Then*

$$\sum_{i=1}^{\infty}\Big(\int_F \varphi_i\,d\mu\Big)^2<\infty. \tag{4.6}$$ [11]

Proof of Lemma 3. The proof can be carried out by using an argument similar to that used in the proof of Lemma 2. For the sake of brevity, set

$$\gamma_i=\int_F \varphi_i\,d\mu \quad (i=1,2,\ldots).$$

Let us start again with the inequality

$$0\leqq\int_F\Big(\lambda-\sum_{i=1}^{n}\gamma_i\varphi_i\Big)^2 d\mu=\lambda^2\mu(F)-2\lambda\sum_{i=1}^{n}\gamma_i^2+$$
$$+\sum_{i=1}^{n}\gamma_i^2\int_F\varphi_i^2\,d\mu+2\sum_{1\leqq i<k\leqq n}\gamma_i\gamma_k\int_F\varphi_i\varphi_k\,d\mu,$$

where λ is a parameter and n is a fixed positive integer.

The last sum on the right-hand side of this inequality can be estimated as follows. Using the Cauchy inequality we get that

$$S=\sum_{1\leqq i<k\leqq n}\gamma_i\gamma_k\int_F\varphi_i\varphi_k\,d\mu\leqq\big[\sum\gamma_i^2\gamma_k^2\big]^{1/2}\Big[\sum\Big(\int_F\varphi_i\varphi_k\,d\mu\Big)^2\Big]^{1/2}\leqq$$
$$\leqq\Big[\sum_{1\leqq i<k\leqq n}\Big(\int_F\varphi_i\varphi_k\,d\mu\Big)^2\Big]^{1/2}\sum_{i=1}^{n}\gamma_i^2.$$

By virtue of Lemma 2 we have

$$\sum_{1\leqq i<k}\Big(\int_F\varphi_i\varphi_k\,d\mu\Big)^2\leqq\int\chi_F^2\,d\mu=C\mu(F),$$

which, combined with the preceding inequality, gives that

$$S\leqq C^{1/2}\mu^{1/2}(F)\sum_{i=1}^{n}\gamma_i^2.$$

Hence we find that

$$0\leqq\lambda^2\mu(F)-2\lambda\sum_{i=1}^{n}\gamma_i^2+\sum_{i=1}^{n}\gamma_i^2\int_F\varphi_i^2\,d\mu+2S\leqq$$
$$\leqq\lambda^2\mu(F)-2\Big(\lambda-\frac{1}{2}K^{1/2}\mu^{1/2}(F)-C^{1/2}\mu^{1/2}(F)\Big)\sum_{i=1}^{n}\gamma_i^2,$$

[11]) Lemma 3 is true for any square integrable function whose support is of finite measure instead of the characteristic function χ_F of the set F.

where we took into account that by (1.5)

$$\int_F \varphi_i^2\, d\mu \leqq \Big(\int_F \varphi_i^4\, d\mu\Big)^{1/2} \Big(\int_F d\mu\Big)^{1/2} \leqq K^{1/2}\mu^{1/2}(F).$$

Choosing

$$\lambda = (K^{1/2}+2C^{1/2})\mu^{1/2}(F),$$

we get that

$$\sum_{i=1}^{n} \gamma_i^2 \leqq \lambda\mu(F) = (K^{1/2}+2C^{1/2})\mu^{3/2}(F),$$

and letting $n\to\infty$, we obtain (4.6), which was to be proved.

§ 5. Proofs of Theorems 2—4

Using Lemma 2 and Lemma 3, the proofs of our converse theorems follow a standard way.

Proof of Theorem 2. We start with the inequality

$$\int_F \Big(\sum_{i=n_0}^{n} c_i\varphi_i\Big)^2 d\mu = \sum_{i=n_0}^{n} c_i^2 \int_F \varphi_i^2\, d\mu + 2 \sum_{n_0\leqq i<k\leqq n} c_i c_k \int_F \varphi_i\varphi_k\, d\mu, \tag{5.1}$$

where n_0 will be determined later. As for the first sum on the right-hand side of (5.1), by (1.7) we have

$$K_1 \sum_{i=n_0}^{n} c_i^2 \leqq \sum_{i=n_0}^{n} c_i^2 \int_F \varphi_i^2\, d\mu \leqq K_2 \sum_{i=n_0}^{n} c_i^2. \tag{5.2}$$

Let us estimate the second sum on the right-hand side of (5.1). Using the Cauchy inequality, the modulus of this sum does not exceed

$$2\Big[\sum_{n_0\leqq i<k\leqq n} c_i^2 c_k^2\Big]^{1/2}\Big[\sum_{n_0\leqq i<k\leqq n} \gamma_{ik}^2\Big]^{1/2} \leqq 2\sum_{i=n_0}^{n} c_i^2 \Big[\sum_{n_0\leqq i<k} \gamma_{ik}^2\Big]^{1/2}, \tag{5.3}$$

where

$$\gamma_{ik} = \int_F \varphi_i\varphi_k\, d\mu = \int \chi_F\varphi_i\varphi_k\, d\mu \quad (i\neq k).$$

Since the characteristic function χ_F is square integrable, F being of finite measure, in virtue of Lemma 2 there exists an integer n_0 such that

$$\sum_{n_0\leqq i<k} \gamma_{ik}^2 < \frac{1}{4}\delta^2 K_1^2 \leqq \frac{1}{4}\delta^2 K_2^2. \tag{5.4}$$

Hence if $n\geqq n_0$, from (5.1)—(5.4) we can conclude inequality (1.8), which was to be proved.

Proof of Theorem 3. We may suppose that E is a set of finite measure.[12]) By (1.11) there is a $K_1^*>0$, which can be taken, e.g., $\frac{1}{2}\liminf_{i\to\infty}\int_E \varphi_i^2 d\mu$, and a positive integer i_1 for which

$$\int_E \varphi_i^2\, d\mu \geqq K_1^* \quad (i > i_1). \tag{5.5}$$

The hypothesis is that for almost every x in E each of the series $\sum_n \alpha_{mn} s_n$ converges to a sum t_m $(m=1, 2, \ldots)$, which tends to a finite limit or, more generally, bounded as $m\to\infty$. Therefore, we can find a subset F of E with $\mu(F)>0$ and a positive number M such that

$$|t_m(x)| \leqq M \quad (x\in F;\ m = 1, 2, \ldots), \tag{5.6}$$

and, in addition, the relation

$$\int_F \varphi_i^2\, d\mu \geqq K_1 \quad (i > i_1) \tag{5.7}$$

also holds. The latter relation readily follows from (5.5) if $\mu(E\setminus F)$ is sufficiently small, because

$$\int_F \varphi_i^2\, d\mu = \int_E \varphi_i^2\, d\mu - \int_{E\setminus F} \varphi_i^2\, d\mu \geqq K_1^* - K^{1/2}\mu^{1/2}(E\setminus F),$$

where we used (1.5) and the Buniakowskii—Schwarz inequality.

Firstly we deal with the case when the summation matrix T^* is row-finite. We apply Theorem 2 with $\delta=\frac{1}{2}$. Then there exists an integer n_0 $(\geqq n_1)$ such that (1.8) holds for every $n\geqq n_0$. Using the elementary inequality

$$(a+b)^2 \geqq \frac{1}{2}a^2 - b^2,$$

we get that

$$\int_F t_m^2\, d\mu \geqq \frac{1}{2}\int_F \left(\sum_{i=n_0}^{\infty} R_{mi} c_i \varphi_i\right)^2 d\mu - \int_F \left(\sum_{i=1}^{n_0-1} R_{mi} c_i \varphi_i\right)^2 d\mu, \tag{5.8}$$

where the sum $\sum_{i=n_0}^{\infty} R_{mi} c_i \varphi_i$ now has only a finite number of terms different from zero. According to (1.8) we have

$$\int_F \left(\sum_{i=n_0}^{\infty} R_{mi} c_i \varphi_i\right)^2 d\mu \geqq \frac{1}{2} K_1 \sum_{i=n_0}^{\infty} R_{mi}^2 c_i^2. \tag{5.9}$$

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[12]) Namely, let $E=\bigcup_{i=1}^{\infty} E_i$, where $\mu(E_i)<\infty$ $(i=1, 2, \ldots)$. If relation (1.11) is not true for any E_i, then, using the Cantor diagonal process, one can easily show that (1.11) is not true for E, either.

The second integral on the right-hand side of (5.8) can be estimated by using Minkowski's inequality as follows:

$$\int_F \left(\sum_{i=1}^{n_0-1} R_{mi} c_i \varphi_i\right)^2 d\mu \leqq \left[\sum_{i=1}^{n_0-1} |R_{mi}||c_i| \left(\int_F \varphi_i^2 \, d\mu\right)^{1/2}\right]^2 \leqq \left[\sum_{i=1}^{n_0-1} |R_{mi}||c_i| K_2^{1/2}\right]^2,$$

where we took into consideration that by (1.5)

$$\int_F \varphi_i^2 \, d\mu \leqq \left[\int_F \varphi_i^4 \, d\mu \int_F d\mu\right]^{1/2} \leqq [K\mu(F)]^{1/2} = K_2.$$

By virtue of (1.12) the inequality $|R_{mi}| \leqq 2$ holds for $i=1, 2, \ldots, i_0-1$ if m is large enough. Therefore, continuing the above argument, for such m's we have

$$\int_F \left(\sum_{i=1}^{n_0-1} R_{mi} c_i \varphi_i\right)^2 d\mu \leqq 4K_2 \left(\sum_{i=1}^{n_0-1} |c_i|\right)^2 = C. \tag{5.10}$$

Collecting (5.6), (5.8), (5.9), and (5.10) we obtain that

$$M^2 \mu(F) \geqq \int_F t_m^2 \, d\mu \geqq \frac{1}{2} K_1 \sum_{i=n_0}^{\infty} R_{mi}^2 c_i^2 - C.$$

Making here $m \to \infty$ and observing (1.12) we get the required result: $\sum c_i^2 < \infty$.

Now we remove the constraint on T^* to be row-finite. This can be done in the same way as in Zygmund's book [16, p. 205]. For the sake of completeness we give the proof here.

Let t_m^* be an expression analogous to t_m, except that the upper limit of summation is not ∞ but a number $N = N(m)$:

$$t_m^* = \sum_{n=1}^{N} \alpha_{mn} s_n .$$

We take N so large that the following conditions be satisfied:

(i) $$|t_m(x) - t_m^*(x)| \leqq \frac{1}{m} \quad \text{for} \quad x \in F \setminus F_m,$$

where

$$\mu(F_m) \leqq \frac{1}{2^{m+1}} \mu(F);$$

(ii) $$\lim_{m\to\infty} \sum_{n=1}^{N} \alpha_{mn} = 1.$$

Setting

$$F^* = \bigcup_{m=1}^{\infty} F_m,$$

we have

$$\mu(F^*) < \mu(F)$$

and on the set $F \setminus F^*$, which is of positive measure, the mean $t_m^*(x)$ tends to a finite limit or is bounded as $m \to \infty$, respectively. But condition (ii) ensures that the t_m^*'s are T^* means corresponding to a row-finite matrix. Thus the general case is reduced to the special case already dealt with.

This completes the proof of Theorem 3.

Proof of Theorem 4. The proof closely follows that of a similar theorem concerning lacunary trigonometric series in Zygmund's book [16, pp. 205—206].

In the course of the proof we assume that $c_i=0$ for some i, say $i<n_0$, where n_0 is determined by Theorem 2, since we may always omit a finite number of terms of $\sum c_i \varphi_i$ without influencing its T^* summability (although this can affect the value of the upper or lower bound of the T^* means).

Set

$$\Gamma_m^2 = \sum_{i=1}^{\infty} R_{mi}^2 c_i^2 \quad (m = 1, 2, \ldots).$$

Suppose that we have (1.13) for every $x \in E$, $\mu(E)>0$, and that $\sum c_i^2$ diverges. Given any positive number ε, there exist an integer m_0 and a set $F \subset E$ with $\mu(F) \geqq \frac{1}{2}\mu(E)$ such

$$t_m(x) \leqq \varepsilon \Gamma_m \quad (x \in F;\ m \geqq m_0).$$

Then

$$\int_F |t_m|\, d\mu \leqq \int_F \{|t_m - \varepsilon\Gamma_m| + \varepsilon\Gamma_m\}\, d\mu = \tag{5.11}$$

$$= \int_F \{2\varepsilon\Gamma_m - t_m\}\, d\mu = 2\varepsilon\mu(F)\Gamma_m - \int_F t_m\, d\mu.$$

We are going to estimate the last integral on the right-hand side by applying Lemma 3. By the Cauchy inequality we get

$$\int_F t_m\, d\mu = \sum_{i=n_0}^{\infty} R_{mi} c_i \int_F \varphi_i\, d\mu \leqq \left[\sum_{i=n_0}^{\infty} R_{mi}^2 c_i^2\right]^{1/2} \left[\sum_{i=n_0}^{\infty} \left(\int_F \varphi_i\, d\mu\right)^2\right]^{1/2} \leqq \varepsilon \Gamma_m, \tag{5.12}$$

if $c_i=0$ for $i<n_0$ and n_0 is chosen so that

$$\sum_{i=n_0}^{\infty} \left(\int_F \varphi_i\, d\mu\right)^2 \leqq \varepsilon^2.$$

This is possible because of (4.6).

Therefore, the right-hand side of (5.11) is less than $2\varepsilon\mu(F)\Gamma_m + \varepsilon\Gamma_m$. This shows that

$$\int_F |t_m|\, d\mu = o(\Gamma_m) \quad (m \to \infty). \tag{5.13}$$

On the other hand, consider the inequality

$$\int_F t_m^2\,d\mu \leqq \Big[\int_F |t_m|\,d\mu\Big]^{2/3}\Big[\int_F t_m^4\,d\mu\Big]^{1/3},$$

which is an immediate consequence of Hölder's inequality. By virtue of Theorem 2, the left-hand side here exceeds some fixed multiple of Γ_m^2. On account of Theorem 1 the integral $\int_F t_m^4\,d\mu$ ($\leqq \int t_m^4\,d\mu$) does not exceed some fixed multiple of Γ_m^4. Thus, $\int_F |t_m|\,d\mu$ exceeds some fixed multiple of Γ_m. This contradicts (5.13) and proves Theorem 4.

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